Area of a cellular complex in a hyperbolic 3-manifold

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Abstract

We give upper bounds on the area of certain 2-dimensional cellular complexes geodesically embedded in hyperbolic 3-manifolds.

Keywords: hyperbolic 3-manifold, cut locus, Dirichlet polyhedron.

1 Introduction

In this paper, we consider a 2-dimensional cellular complex geodesically embedded in a hyperbolic 3-manifold M and study its *area* defined as the sum of the area of its 2-cells. Such an object, for instance, comes from the boundary of a Dirichlet polyhedron, called *Dirichlet complex*, if M is closed. When M has totally geodesic boundary ∂M, another example is given by the *cut locus* with respect to ∂M. Both are often used to study hyperbolic 3-manifolds and kleinian groups, and, in this paper, we give upper bounds on the area of such complexes if they are generic.

First we consider the case where M is a closed hyperbolic 3-manifold. As usual, we identify the universal cover of M with 3-dimensional hyperbolic space $\mathbb{H}^3$. Fix a point x in M and a lift $\tilde{x}$ of x in $\mathbb{H}^3$. Then the *Dirichlet complex* D_x of M (with center x) is defined as the set of points in $\mathbb{H}^3$ closer to $\tilde{x}$ than to $\gamma\tilde{x}$ for any $\gamma \in \Gamma$, where Γ denotes the covering transformation group. This becomes a convex fundamental polyhedron for Γ with a finite number of totally geodesic sides (see [6] for example). The image of the boundary ∂D_x under the covering projection gives a 2-dimensional cellular complex geodesically embedded in M, which we call a *Dirichlet complex* with respect to x.

A Dirichlet polyhedron D_x is called *generic* if the polyhedral decomposition of $\mathbb{H}^3$ obtained from all translates of D_x under Γ is dual to some triangulation of $\mathbb{H}^3$. We say that a Dirichlet complex is *generic* if the corresponding Dirichlet polyhedron is generic.

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Then our first theorem is the following:

Theorem 1.1. *Let M be a closed hyperbolic 3-manifold and C a Dirichlet complex in M. Suppose that C is generic. Then $Area(C) < \pi(v-2)$ holds, where v denotes the number of the 0-cells of C.*

The following is an immediate corollary of the theorem above.

Corollary 1.2. *Let M be a closed hyperbolic 3-manifold and D_x a Dirichlet polyhedron of M. Suppose that D_x is generic. Then $Area(\partial D_x) < 2\pi(v/4-2)$ holds, where v denotes the number of the vertices of D_x.*

proof. Let C be the generic Dirichlet complex in M corresponding to D_x. Note that each two faces of D_x are glued together in M by covering projection. Thus $Area(\partial D_x)=2Area(C)$ holds. Also note that each four vertices of D_x are glued together in M since D_x is generic. Thus the number of the vertices of D_x is equal to that of the 0-cells of C multiplied by four. Therefore this corollary follows from Theorem 1.1. □

Remark that it was shown in [4] that Dirichlet polyhedra are generic for almost all points in M.

Also, by definition, we obtain the following immediately.

Corollary 1.3. *For every point x in a closed hyperbolic 3-manifold, there exists a point y arbitrary close to x such that there exist more than two points each of which is connected by four distinct minimal geodesic segments to y.* □

The existence of such points in a compact, non-positively curved Riemannian manifold is known [3] and generically these are only finite many. In general, there exist uncountably many points each of which is connected by at most three distinct geodesic segments to the given point.

Next, in the case that M has non-empty totally geodesic boundary ∂M, we consider the *cut locus* (with respect to ∂M), which is defined as the set of points in M admitting at least two distinct shortest paths to ∂M. This becomes a geodesic, convex 2-dimensional cellular complex embedded in M. See [5]. Remark that the *canonical decomposition* of M is defined as the geometric dual to C in [5]. Then we obtain the following:

Theorem 1.4. *Let M be a compact hyperbolic 3-manifold with non-empty totally geodesic boundary ∂M and C the cut locus with respect to ∂M. Suppose that the canonical decomposition of M consists of t truncated tetrahedra. Then $Area(C) < t\pi + 1/2\ Area(\partial M)$ holds. Moreover $Area(C) < 3\pi t$ holds.*

2 Area of geodesic cellular complex

We start with an observation about the area $Area(C)$ of a *hyperbolic* 2-dimensional cellular complex C. We say that C is *hyperbolic* if there are identifications of closed 2-cells of C with convex polygons in $\mathbb{H}^2$ which match isometrically along 1-cells of C.

Let n_i be the number of the i-cells of C for $i = 0, 1, 2$ and $\sigma_1, \sigma_2, \ldots, \sigma_{n_2}$ 2-cells of C. Then $Area(C) = \sum_{i=1}^{n_2} Area(\sigma_i)$ holds. It follows that

$$\begin{aligned} Area(C) &= \sum_{i=1}^{n_2}((m_i - 2)\pi - \theta_i) \\ &= \left(\sum_{i=1}^{n_2} m_i\pi\right) - 2n_2\pi - \left(\sum_{i=1}^{n_2}\theta_i\right), \end{aligned}$$

where m_i denotes the number of internal angles of σ_i and θ_i the sum of the internal angles of σ_i for $1 \leq i \leq n_2$.

Moreover, if C is *generic*, in the sense that the link of each 0-cell is isomorphic to the complete graph of order 4, then we have $\Sigma_{i=1}^{n_2} m_i = 3n_1$. Thus we obtain

$$Area(C) = (3n_1 - 2n_2)\pi - \sum_{i=1}^{n_2}\theta_i. \tag{1}$$

Now we consider a 2-dimensional cellular complex C *geodesically* embedded in a hyperbolic 3-manifold M. That is, each 1-cell and 2-cell of C is a geodesic segment and a totally geodesic polygon in M respectively. We will always assume that such a complex is *convex*, i.e., each closed 2-cell is a convex polygon in M. Obviously it can be regarded as a hyperbolic 2-dimensional cellular complex.

Among such cellular complexes, as we stated in the previous section, we will focus on some particular ones, which is shown to have the following geometric properties. We say that a 2-dimensional cellular complex C geodesically embedded in M is *global in the link of each point* if any ball neighborhood of each point in C with boundary S satisfies that the intersection $S \cap C$ is not contained in any open hemisphere of S.

In this case, we have the following estimate on the area.

Theorem 2.1. *Let M be a hyperbolic 3-manifold and C a generic 2-dimensional cellular complex geodesically embedded in M. Suppose that C is global in the link of each point. Then we have*

$$-2\pi\chi(C) \leq Area(C) < n_0\pi - 2\pi\chi(C),$$

where $\chi(C)$ denotes the Euler characteristic, $Area(C)$ the area and n_0 the number of 0-cells of C.

Remark that the inequality still holds if C is an image of a cell-wise embedding instead of an embedding.

To prove Theorem 2.1, we prepare a lemma about an embedding of the complete graph K_4 of order 4 into the 2-dimensional sphere S^2. In the following, S^2 is assumed to have the Riemannian metric of constant curvature +1.

As usual, we regard a finite graph as a 1-dimensional cellular complex by setting a vertex as a 0-cell and an edge as a closed 1-cell. Given an embedding f of a finite graph into a surface, its image G is naturally identified with the original graph, and so the image of a vertex and an edge by f is said to be a vertex and an edge of G.

Lemma 2.2. *Let G be the image of an embedding of K_4 into S^2. Suppose that*

(1) *each edge of G is a shortest geodesic arc on S^2 connecting its endpoints, and*

(2) *G is not contained in any open hemisphere.*

Let E be the sum of the lengths of the edges of G. Then $3\pi < \mathrm{E} \leq 4\pi$ holds.

This is an almost immediate consequence of the result obtained by Gaddum in [2]. In the next section, we will give an elementary proof of Lemma 2.2.

Proof of Theorem 2.1. We use the same notations as in the previous observation. Let us consider the small ball neighborhoods for all 0-cells of C, and identify each of their boundaries with the unit 2-sphere S^2. The intersection of such a sphere and C yields a 1-dimensional cellular complex, regarded as a graph, embedded in S^2. In this setting, we note that the sum of lengths of the edges for all such graphs is equal to the sum of internal angles for all 2-cells of C.

Now, the assumption on C guarantees that Lemma 2.2 can be applied to all such graphs. Thus, from Equation (1), we obtain

$$3n_1\pi - 2n_2\pi - 4n_0\pi \leq Area(C) < 3n_1\pi - 2n_2\pi - 3n_0\pi.$$

Since C is generic, we have $n_1 = 2n_0$, and so

$$Area(C) \geq 2n_1\pi - 2n_2\pi - 2n_0\pi = -2\pi\chi(C)$$

and

$$Area(C) < 2n_1\pi - 2n_2\pi - 2n_0\pi + n_0\pi = n_0\pi - 2\pi\chi(C)$$

hold. □

We remark that the results corresponding to Theorem 2.1 can be obtained for spherical manifolds. In this case, all inequalities are reversed. The proof can be done in the same way, and so we do not include it here.

3 Geodesic graph on the 2-sphere

In this section, we give a proof of Lemma 2.2. Let us start with recalling fundamentals of spherical geometry. Let u_1, u_2, u_3 be points on S^2 such that no two of them are antipodal and no great circle includes all the three points. Let Λ_i be the closed hemisphere whose boundary contains the other two points than u_i and whose interior contains u_i for i =1,2,3. The *spherical triangle* Δ with the vertices u_1, u_2, u_3 is defined as the intersection $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3$. Then we have the following:

- Δ is convex, i.e., any pair of points in Δ is connected by a geodesic arc in Δ. Moreover the arc is shortest among the arcs connecting the points, and the length is equal to the spherical distance between the points which is strictly less than π.
- The length of an edge of Δ is less than the sum of the lengths of the other two edges (*the triangle inequality*).

Proof of Lemma 2.2 Let v_1, v_2, v_3, v_4 be the vertices of G. Let e_{ij} denote the edge of G connecting v_i and v_j for $1 \leq i, j \leq 4$. Note that the assumption (1) implies that the length of e_{ij} is equal to the spherical distance d_{ij} on S^2 between v_i and v_j for $1 \leq i, j \leq 4$. Thus it suffice to show that

$$3\pi < \sum_{1\le i<j\le 4} d_{ij} \le 4\pi.$$

In the following, the antipodal point of v_i is denoted by v_{i+4} for $1 \le i \le 4$. Also d_{ij} denotes the spherical distance between v_i and v_j for $1 \le i, j \le 8$.

First we consider the case that a couple of the vertices, say v_1 *and* v_2, are antipodal, equivalently, $d_{12} = \pi$. This implies that $d_{13} + d_{32} = d_{14} + d_{42} = \pi$ holds. Together with $0 < d_{34} \le \pi$, we have $3\pi < \sum_{1\le i<j\le 4} d_{ij} = 4\pi\cdot$

Next consider the case that all the four vertices are contained in a great circle. Suppose for example that v_1, v_2, v_3, v_4 lies in a great circle Γ in this order. Since G is the image of an embedding, the edge e_{13} is not contained in Γ. This implies that e_{13} is a half of a great circle and $d_{13} = \pi$. Also we see that $d_{24} = \pi$ and so we obtain $\sum_{1\le i<j\le 4} d_{ij} = 4\pi$.

Thus, in the following, we assume that $d_{ij} \ne \pi$ for $1 \le i, j \le 4$ and at most three vertices of G lie on a great circle.

Next consider the case that three vertices are contained in a great circle. Suppose for example that v_1, v_2 and v_3 lie on a great circle. Then, by the triangle inequality, we have $d_{41} + d_{42} > d_{12}$, $d_{42} + d_{43} > d_{23}$ and $d_{43} + d_{41} > d_{31}$. These are added to obtain

$$2(d_{41} + d_{41} + d_{41}) > d_{12} + d_{23} + d_{31} = 2\pi.$$

Thus

$$\sum_{1\le i<j\le 4} d_{ij} = (d_{41} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} > \pi + 2\pi = 3\pi.$$

In the same way as above, we have $d_{45} + d_{46} + d_{47} > \pi$. Since $d_{4j} = \pi - d_{4(j+4)}$ for $j = 1, 2, 3$,

$$\begin{aligned}\sum_{1\le i<j\le 4} d_{ij} &= (d_{41} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} \\ &= 3\pi - (d_{45} + d_{46} + d_{47}) + d_{12} + d_{23} + d_{31} \\ &< 3\pi - \pi + 2\pi = 4\pi\end{aligned}$$

holds.

Finally we consider the case that the four vertices are in a general position: We assume that $d_{ij} \ne \pi$ for $1 \le i, j \le 4$ and at most two vertices of G lie on a great circle. This means that for any three of the points there is a triangular face which includes the three points as vertices.

Then, by the triangle inequality, we have $d_{53} + d_{63} > d_{56}$, $d_{54} + d_{64} > d_{56}$, and $d_{53} + d_{54} > d_{34}$ and $d_{63} + d_{64} > d_{34}$. Add these to obtain

$$d_{53} + d_{63} + d_{54} + d_{64} > d_{34} + d_{56}.$$

Here note that $d_{ij} = \pi - d_{(i-4)j}$ for $i = 5, 6$, $j = 1, 2, 3$, and $d_{56} = d_{12}$. These imply that

$$4\pi - (d_{13}+d_{23}+d_{14}+d_{24}) > d_{34}+d_{12}.$$

Consequently we have

$$4\pi > \sum_{1\leq i<j\leq 4} d_{ij}.$$

In the following, let Δ be the spherical triangle bounded by e_{12}, e_{23} and e_{31}.

Claim 1. *The antipodal point* v_8 of v_4 *is included in the interior* of Δ.

Proof. Let Γ_i be the great circle including an edge of Δ but not including v_i for $i = 1, 2, 3$. By the assumption above, v_4 and hence v_8 never lie on $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Note that $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ decomposes S^2 into eight spherical triangles.

Assume for a contradiction that v_8 is not included in the interior of Δ. Then v_4 is included in the interior of one of the seven spherical triangles other than the antipodal image of Δ. This implies all the four points v_1, v_2, v_3 and v_4 are included in the closed hemisphere bounded by one of Γ_1, Γ_2 or Γ_3. Since the four vertices are assumed in a general position, there is an open hemisphere which contains whole G. This contradicts the assumption (2) of the lemma. □

Claim 2. *The inequality* $d_{12} + d_{23} + d_{13} > d_{81} + d_{82} + \mathrm{d}_{83}$ *holds.*

Proof. Since the length of each edge is less than π, and by Claim 1, the edge e_{13} intersects the great circle including v_2 and v_8 at just one point v_9. Let d_{i9} or d_{9i} denote the distance between v_i and v_9 for $1 \leq i \leq 8$. The distance d_{19} is realized by a geodesic arc included in e_{13} and also is d_{93}. Thus $d_{13} = d_{19} + d_{93}$ holds.

The distance d_{29} is realized by a geodesic arc e_{29} in Δ since Δ is convex. In particular, the arc e_{29} contains v_8 and so $d_{29} = d_{28} + d_{89}$ holds.

Together with the triangle inequality $d_{12} + d_{19} > d_{29}$ and $d_{98} + d_{93} > d_{83}$, we conclude

$$d_{12} + d_{13} = d_{12} + d_{19} + d_{93} > d_{29} + d_{93} = d_{28} + d_{89} + d_{93} > d_{28} + d_{83}.$$

In the same way, we have $d_{21} + d_{23} > d_{81} + d_{83}$ and $d_{31} + d_{32} > d_{81} + d_{82}$.

By adding these inequalities, we obtain

$$d_{12}+d_{23}+d_{31} > d_{81}+d_{82}+d_{83}.$$

□

Together with the equations $d_{8j} = \pi - d_{4j}$ for $j = 1, 2, 3$, we conclude that

$$\sum_{1 \le i < j \le 3} d_{ij} > 3\pi - \sum_{1 \le k \le 3} d_{k4} .$$

This completes the proof. $\square$

4 In a hyperbolic 3-manifold

Here we give proofs of the theorems stated in the first section.

4.1 Boundary of Dirichlet polyhedron

Here let M be a closed hyperbolic 3-manifold and regard $\mathbb{H}^3$ as the universal cover of M.

Proof of Theorem 1.1 Let p: $\mathbb{H}^3 \to M$ be the covering projection and D_x a generic Dirichlet polyhedron such that $C = p(\partial D_x)$. Since D_x is a convex polyhedron, $M - C$ is locally convex at C, which means that C is global in the link of each point. By definition, if D_x is generic, then C is also. Thus we can apply Theorem 2.1 to this C, and then $Area(C) < v\pi - 2\pi\,\chi(C)$ is obtained. Note that $\chi(C) = \chi(M) + 1 = 1$ holds, where $\chi(M)$ denotes the Euler characteristic of M, which is 0. This completes the proof of Theorem 1.1. $\square$

4.2 Cut locus

Next let M be a compact hyperbolic 3-manifold with non-empty totally geodesic boundary ∂M. Then the cut locus C with respect to ∂M is shown to be geodesic, convex and global in the link of each point as follows.

Lemma 4.1. *Let M be a compact hyperbolic 3-manifold with non-empty totally geodesic boundary ∂M and C the cut locus with respect to ∂M. Then C is geodesic, convex and global in the link of each point.*

Proof. We can assume that the universal cover of M is a subspace embedded in $\mathbb{H}^3$. Let $\widetilde{C}$ be the preimage of C by the covering projection. Take a connected component N of $\mathbb{H}^3 - \widetilde{C}$ and let P be the intersection of N with the preimage $\widetilde{\partial M}$ of ∂M.

Consider the middle fences of the short cuts between P and any other components of $\widetilde{\partial M}$. Then N appears as the intersection of the half spaces bounded by such middle fences, that is, N is a convex polyhedron in $\mathbb{H}^3$. This implies that each face of N, which is isometric to a 2-cell of C, is a convex totally geodesic polygon. Also this means the complement of C in M is locally convex, and it implies that C is global in the link of each point. $\square$

Proof of Theorem 1.4. Let n_i be the number of the i-cells of C for i=0, 1, 2. Recall that the canonical decomposition of M is the geometric dual to C. Moreover the number of 0-cells of C is equal to the number of tetrahedra, that is, $n_0 = t$ holds. Then, by Lemma 4.1, we can apply Theorem 2.1 and obtain $Area(C) < n_0\pi - 2\pi\,\chi(C)$.

Since C is a spine [5], there exists a retraction of M onto C. By using this retraction, a cellular decomposition of ∂M is induced from that of C. Let N_i be the number of the i-cells of ∂M for i=0, 1, 2. Then $n_1 = 2n_0$, $4n_0 = N_0$, $3n_1 = N_1$ and $2n_2 = N_2$ hold in this setting. By direct calculation, we have $2\chi(C) = \chi(\partial M)$, where $\chi(\partial M)$ denotes the Euler characteristic of ∂M. Thus the Gauss-Bonnet Theorem implies $Area(C) < t\pi + 1/2\ Area(\partial M)$.

The last statement of the theorem follows from the next lemma. □

Lemma 4.2. *Let M be a compact hyperbolic 3-manifold with non-empty totally geodesic boundary ∂M. Suppose that the canonical decomposition of M consists of t truncated tetrahedra. Then $Area(\partial M) < 4\pi t$ holds.*

Proof. As we remarked before, $n_1=2n_0$, $4n_0=N_0$, $3n_1=N_1$ and $2n_2=N_2$ hold. Therefore the Euler characteristic χ of ∂M is $4n_0 - 6n_0 + 2n_2 = 2(n_2 - n_0)$. Since n_2 is a positive integer, we have $-\chi < 2n_0$. By the Gauss-Bonnet theorem, we have $Area(\partial M) = -2\pi\chi$, and so $Area(\partial M) < 4\pi n_0 = 4\pi t$ is achieved. □

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